# Quiz 2 solution

# Cmps 211

# Question 2

## Question 2

**(a) problem definition**

input $\{a_j\}_{j=1}^{n}$, '<'

size of input

order is defined for comparison

Output let $P(i) = $ " $a_i < \prod_{k=1}^{i-1} a_k$ "

$\therefore$ return $A = \{a_i \mid P(i), 1 \leq i \leq n\}$

**(b) paradigm + pseudo-code**

Paradigm: iterative

Pseudocode:

```
lessThanProd( {a_j}_{j=1}^n , '<' )
{ A ← Φ ;
  Prod ← 1 ;
  for ( j = 2 ; j <= n; j++)
  {
        prod ← prod * a_{j-1};
        if a_j < prod
             A ← A ∪ {a_j}
  }
  return A
}
```

## (c) Best/worst case definition

Input size : $n$

- Best case : $A = \emptyset \quad \forall_{j \geq 2} \ \neg P(j)$

- worst case : $A = \{a_j\}_{j=2}^{n} : \forall_{j \geq 2} \ P(j)$

## (d) Space requirement

every read/write instruction requires space

$$S(n) = \underbrace{n}_{\text{read input}} + \underbrace{n}_{\text{set A func return}} + \underbrace{1}_{\text{for prod variable}} + \underbrace{1}_{\text{for } j}$$

$= 2n + 2$

$= 2(n+1)$

$= O(n)$

## (e) Worst case runtime

$$T(n) = \underbrace{1}_{\text{assign A}} + \underbrace{1}_{\text{assign prod}} + \underbrace{(n-1)}_{\text{\# of iterations}}\underbrace{(5)}_{\text{\# cost per iteration}} + \underbrace{1}_{\text{return A}}$$

$= 2 + 5n - 5 + 1$

$= 5n - 2$

(f) order of growth

Proof

$$5n - 2 = O(n)$$
$$= \Theta(n)$$

(g) prove (P)

$$5n - 2 = \Theta(n)$$

Prove $\exists\, c_1, c_2, n_0 : \quad c_1 n \leq 5n-2 \leq c_2 n \quad \forall n \geq n_0$

$$\boxed{\begin{array}{l} n_0 = 1 \\ c_1 = 1 \\ c_2 = 7 \end{array}} \quad 1n \leq 5n - 2 \leq 7n$$

$\therefore\ 5n - 2 = \Theta(n)$

(h) loop invariants

after the end of $j^{th}$ iteration set A contains those elements $\leq$ product of previous elements seen so far

"After the $j^{th}$ iteration

$$Q(j) = A = \{ a_j \mid P(i) : 1 \leq i < j \}"$$

index of iteration

(i) ~~Prove~~ loop invariant

by Induction

Base cases: ~~Q(1)~~ assert $Q(1)$

$Q(1)$ = at end of $1^{st}$ iteration $A = \{ a_j \mid P(i) : 1 \le i < 2 \}$

before $1^{st}$ loop we know

$prod = 1$
$A = \emptyset$

during if ~~prod~~ $a_2 < prod * a_1$
$A = \{a_2\}$
else
$A = \emptyset$

After
$A = \emptyset \vee A = \{a_2\}$ $\therefore$ $Q(1)$

Inductive Step:

assert $Q(j)$ assuming $Q(j-1)$

before
assume $Q(j-1)$:

at $j^{th}-1$ iteration $A = \{ a \mid P(i) : 1 \le i < j-1 \}$

during if $a_{j-1} < prod * a_{j-2}$
$A = A \cup \{a_{j-1}\}$
else
$A = A$

after

$A = A \vee A \cup \{a_{j-1}\}$ $\therefore$ $Q(j)$